Polarization matrix and geometric phase

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Following Jones calculus, we have evaluated the polarization matrix of an optical system whose two orthonormal eigenvectors are represented by two-component spinors of spherical harmonics. In addition, we have studied the geometric phase of polarized light whose plane of polarization is rotated over a closed path by a rotator. [S1063-651X(97)08707-2]

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I. INTRODUCTION

The cyclic variation of external parameter often leads to a net evolution involving a phase associated with the geometry of the path traversed in the parameter space. For noncyclic evolution, this phase can be written as the function of the end points of the path traversed. This is the well-known geometric phase abound in many areas of physics. First, it was discovered in optics in the 1930s [1]. Later, in the mid-1950s, Pancharatnam [2] introduced the idea of such a phase in the study of the classical theory of polarized light on the Poincaré sphere. After the discovery of Berry's geometric phase [3] in 1984, Nityananda and Ramseshan pointed out its resemblance to the Pancharatnam phase [4]. Later on, Berry discussed the connection of his phase [5] with that of Pancharatnam, considering photons as two-component spinors. Several experiments have been performed to demonstrate the geometric phase (GP) in optics [6], in which Bhandari and Samuel [7] reported first a direct experimental observation of the Pancharatnam-Berry phase in the nonunitary evolution on the Poincaré sphere by means of a laser interferometer.

The study of optical polarization has been enriched by the valuable works of Bhandari [8] as well as Simon and Mukunda [9] and also Simon, Mukunda, and Sudarshan [10] from group-theoretical and geometrical aspects. Recently, we have found a complete quantum illustration of the GP in optics from the work of Klyshko [11].

In light of Berry's work, we have recently calculated relativistically [12] the Berry phase of plane-polarized light over a triangle on the Poincaré sphere whose vertices express three different nonorthogonal polarizations, respectively. It has been assumed there that as light passes through an anisotropic medium the photon fixes its helicity. If positive helicity corresponds to right circular polarization, then negative helicity belongs to left circular polarization. In analogy with a spin system, we may suggest that a photon with a fixed helicity can be viewed as if a *direction vector* y_μ is attached at the space-time point x_μ in Minkowski space, so that we can write the coordinate in the complexified space-time as $z_\mu = x_\mu + i y_\mu$. Here the additional condition is $|y_\mu|^2 = 0$, which ensures the masslessness of photon. Introducing the spinorial variable $\theta(\bar{\theta})$, (through y_μ) in the space-time geometry, we finally express the polarized photon as a two-component spinor of spherical harmonics.

In this paper, using Jones calculus, we shall construct the polarization matrix of the optical system in the relativistic framework. From this formulation, we shall also aim at studying the geometric phase of polarized light (circular, linear) whose plane of polarization suffers a rotation over a closed path by a rotator.

II. SPINORIAL REPRESENTATION OF POLARIZED PHOTONS

A light beam is said to be polarized [13] whenever it is transmitted through a certain crystalline medium that allows electrical anisotropy. This indicates that the photon in the polarized beam fixes its helicity whose direction changes with the change of plane of polarization. In an anisotropic space a particle having a fixed helicity can be viewed as if a *direction vector* is attached at the space-time point [14]. From relativistic point of view, if x_μ is the mean position of the particle and y_μ indicates the direction vector, then we can consider the resultant coordinate in the complexified space as $z_\mu = x_\mu + i y_\mu$. This extended structure indicates that the acquirance of mass and the masslessness condition is achieved when we have $|y_\mu|^2 = 0$. It can be shown that the two opposite orientations of the direction vector represent two *internal helicities* corresponding to fermions and antifermions. In view of this, we can formulate the *internal helicity* in terms of two-component spinorial variables $\theta(\bar{\theta})$.

Indeed, in the complexified space-time, we can write the chiral coordinate as

$$z_\mu = x_\mu + (i/2)\lambda_\mu{}^\alpha \theta_\alpha, \tag{1}$$

where we have taken

$$y_\mu = (1/2)\lambda_\mu{}^\alpha \theta_\alpha, \tag{2}$$

θ_α ($\alpha = 1,2$) being a two-component spinor. If we now replace the chiral coordinates by their matrix representation

$$z^{AA'} = x^{AA'} + (i/2)\lambda_\alpha{}^{AA'} \theta^\alpha, \tag{3}$$

where

$$x^{AA'} = \frac{1}{\sqrt{2}} \begin{bmatrix} x^0 - x^1 & x^2 + i x^3 \\ x^2 - i x^3 & x^0 + x^1 \end{bmatrix}, \tag{4}$$

with

$$\lambda_\alpha{}^{AA'} \in \mathrm{SL}(2,c),$$

we find the helicity operator

$$S_{\rm hel}=-\lambda_{\alpha}{}^{AA'}\theta^{\alpha}\overline{\pi_A}\pi_{A'}\,, \tag{5}$$

which we identify as the *internal helicity* and corresponds to the fermion number when the two opposite orientations of *internal helicities* represent particle and antiparticle. It may be noticed that we have taken the matrix representation of p_μ (conjugate to x_μ in the complex coordinate $z_\mu=x_\mu+iy_\mu$) as $p^{AA'}=\overline{\pi^A}\pi^{A'}$, implying $p_\mu^2=0$. So the particle will have its mass due to the nonvanishing character of the quantity y_μ^2. It is observed that the complex conjugate of the chiral coordinate (1) will give rise to a massive particle with opposite *internal helicity* corresponding to an antifermion. In the null plane where $y_\mu^2=0$, we can write the chiral coordinate for massless spinor as

$$z^{AA'}=x^{AA'}+\frac{i}{2}\,\overline{\theta^A}\theta^{A'}, \tag{6}$$

where the coordinate y^μ is replaced by $y^{AA'}=(1/2)\overline{\theta^A}\theta^{A'}$. In this case the helicity operator is given by

$$S=-\overline{\theta^A}\theta^{A'}\overline{\pi_A}\pi_{A'}=-\overline{\varepsilon}\varepsilon, \tag{7}$$

where $\varepsilon=i\theta^{A'}\pi_A$ and $\overline{\varepsilon}=-i\theta^A\pi_{A'}$. The corresponding twistor equation describes a massless spinor field. In case of massive spinor, we can define a plane D^-, where, for coordinate $z_\mu=x_\mu+iy_\mu$, y_μ belongs to the interior of forward lightcone $(y\gg 0)$ and as such represents the upper half-plane with the condition $\det y^{AA'}>0$ and $\frac{1}{2}{\rm Tr}y^{AA'}>0$. The lower half-plane D^+ is given by the set of all coordinates z_μ with y_μ in the interior of the backward lightcone $(y\ll 0)$. The map $z\rightarrow z^*$ sends the upper half-plane to the lower half-plane. The space M of null planes ($\det y^{AA'}=0$) is the Shilov boundary so that a function holomorphic in $D^-(D^+)$ is determined by its boundary values. Thus, if we consider that any function $\phi(z)=\phi(x)+i\phi(y)$ is holomorphic in the whole domain, the helicity $+\frac{1}{2}(-\frac{1}{2})$ in the null plane may be taken to be the limiting value of the *internal helicity* in the upper (lower) half-plane. Thus massless spinor exists in this plane.

To explain our problem in terms of spinorial variables, the photon with a fixed helicity in the polarized light can be viewed as a massless spinor with helicity $+1/2$ or $-1/2$ on the Shilov boundary. If a photon with a massless $+1$ helilcity state is associated with a fermion having helicity $+1/2$, then the helicity -1 can be associated with an antifermion with helicity $-1/2$.

It may be noted that the wave function $\phi(z_\mu)=\phi(x_\mu)+i\phi(y_\mu)$ can be treated to describe a particle moving in the external space-time having the coordinate x_μ with an attached *direction vector* y_μ. Thus the wave function should take into account the polar coordinates r, θ, ϕ along with the angle χ which specify the rotational orientation around *direction vector* y_μ. For an extended particle θ, ϕ, and χ just represent the three Euler angles [15].

In a three-dimensional (3D) anisotropic space, we can consider an axisymmetric system where the anisotropy is introduced along a particular direction. It is to be noted that in this anisotropic space, the components of the linear momentum satisfy a commutation relation of the form

$$[p_i,p_j]=i\mu\epsilon_{ijk}\frac{x^k}{r^3}. \tag{8}$$

In such a space, the conserved angular momentum J is represented by

$$\vec{J}=\vec{r}\times\vec{p}-\mu\vec{r}. \tag{9}$$

It follows that $J^2=L^2-\mu^2$ instead of L^2 is a conserved quantity. In general, μ, which is the measure of the anisotropy, is given by the eigenvalue of the operator $i\delta/\delta x$ and can take the values $\mu=0,\ \pm 1/2,\ \pm 1,\ldots$. The spherical harmonics incorporating the term μ have been extensively studied by Fiertz [16] and Hurtz [17]. Following them, we write

$$Y_l^{m,\mu}=(1+x)^{-(m-\mu)/2}(1-x)^{-(m+\mu)/2}\,\frac{d^{l-m}}{d^{l-m}x}$$
$$\times[(1+x)^{l-\mu}(1-x)^{l+\mu}]e^{im\phi}e^{-i\mu\chi}, \tag{10}$$

with $x=\cos\theta$.

In the anisotropic space a fermion with helicity $+1/2$ can be treated as a scalar particle moving with $l=1/2$ with $l_z=+1/2$. The specification of the l_z value for the particle and antiparticle states then depicts it as a chiral spinor. From the relation (10), we can construct the spherical harmonics for $l=1/2$ and $m=\pm 1/2$, $\mu=\pm 1/2$. They are given in terms of the components (θ,ϕ,χ) as follows:

$$Y_{1/2}^{1/2,1/2}=\sin\frac{\theta}{2}\,e^{i(\phi-\chi)/2},$$

$$Y_{1/2}^{-1/2,1/2}=\cos\frac{\theta}{2}\,e^{-i(\phi+\chi)/2},$$

$$Y_{1/2}^{1/2,-1/2}=\cos\frac{\theta}{2}\,e^{i(\phi+\chi)/2},$$

$$Y_{1/2}^{-1/2,-1/2}=\sin\frac{\theta}{2}\,e^{-i(\phi-\chi)/2}. \tag{11}$$

These represent spherical harmonics for half-orbital angular momentum in an anisotropic space, and it is to be noted that from these spherical harmonics we can construct the product wave functions Y_1^1, Y_1^{-1}, Y_1^0 which we have used to represent the polarization matrix in our previous work [12]. It has been shown in [14] that the two-component spinor and its conjugate state can be formed from the above spherical harmonics.

In the next section we shall use the above spherical harmonics to represent the polarized photon and also shall evaluate the polarization matrix M of the optical system.

III. MATRIX FORMULATION OF THE OPTICAL SYSTEM AND THE GEOMETRIC PHASE

The passage of plane-polarized light through a partial polarizer or retardation plate having its plane of polarization parallel to either of the principal axes suffers no change in the state of polarization. Similarly, both types of circularly polarized light acquire no change in their state of polarization in passing through a rotator. This property of light with respect to the optical element was presented in a 2×2 matrix method by Jones [18]. The matrix M^n of the optical system of n components having eigenvectors ε_i satisfies the condition

$$M^n \varepsilon_i = d_i \epsilon_i , \tag{12}$$

where d_i is the constant known as eigenvalue corresponding to the eigenvectors ε_i. For one component optical element the matrix becomes

$$M = \begin{pmatrix} m_1 & m_4 \\ m_3 & m_2 \end{pmatrix}.$$

Jones has pointed out that the matrix M of the optical system is then uniquely determined by the relation

$$M = TDT^{-1}, \tag{13}$$

$$M = \Delta^{-1} \begin{pmatrix} d_1 a_1 b_2 - d_2 a_2 b_1 & -(d_1 - d_2) a_1 a_2 \\ (d_1 - d_2) b_1 b_2 & d_2 a_1 b_2 - d_1 a_2 b_1 \end{pmatrix}, \tag{14}$$

where

$$T = \begin{pmatrix} a_1 & a_2 \\ b_1 & b_2 \end{pmatrix}, \tag{15}$$

$$T^{-1} = \Delta^{-1} \begin{pmatrix} b_2 & -a_2 \\ -b_1 & a_1 \end{pmatrix}, \tag{16}$$

having

$$\Delta = a_1 b_2 - a_2 b_1 ,$$

$$D = \begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix},$$

and with no loss in generality the orthonormal eigenvectors are

$$\varepsilon_1 = \begin{pmatrix} a_1 \\ b_1 \end{pmatrix}, \tag{17}$$

$$\varepsilon_2 = \begin{pmatrix} -b_1^* \\ a_1^* \end{pmatrix}. \tag{18}$$

The introduction of a rotator along the path of the light beam transforms the polarization matrix M to M' by the relation

$$M' = S(\omega) M S(-\omega), \tag{19}$$

where $S(\omega)$ is the rotation matrix [19],

$$S(\omega) = \begin{pmatrix} \cos\omega & -\sin\omega \\ \sin\omega & \cos\omega \end{pmatrix}. \tag{20}$$

This property of rotation of the plane of polarization over a closed path, will be utilized to find out the geometric phase of polarized light which will be the eigenvector of our polarization matrix.

Using the above calculus of Jones, we want to construct in the relativistic framework the polarization matrix, considering the photons as two-component spinors of spherical harmonics. The similarity between the behavior of polarized photon and spinor is followed by the early work of Pancharatnam [2] where the phase developed by the cyclic change of polarization on the Poincaré sphere becomes

$$\langle A | A' \rangle = \exp(-i \Omega_{ABC}/2).$$

It indicates that for a solid angle 2π, the phase shift between $|A\rangle$ and $|A'\rangle$ is π, which indicates the reverse of the field. Since we are taking here a fixed polarization of a photon, we can consider it as represented by a fermion with a fixed helicity, though fermions change sign in a 2π rotation and photons do not. That means we can consider the photon with helicity $+1$ as a fermion with helicity $+1/2$ and, on the other hand, helicity -1 can be identified as an antifermion with helicity $-1/2$. In fact, in the discussion of the previous section, this helicity plays the key role. It is visualized by the term y_μ in the coordinate $z_\mu = x_\mu + i y_\mu$ of a polarized photon or in $z_\mu = x_\mu - i y_\mu$ of opposite polarization, where masslessness is ensured by the condition $|y_\mu|^2 = 0$. To have an equivalent representation in polar coordinates one needs the angles θ, ϕ, and χ to represent the field function in terms of spherical harmonics $Y_l^{m,\mu}$. In view of this, we want to represent the polarized photon as

$$\varepsilon_1 = \begin{pmatrix} Y_{1/2}^{1/2,1/2} \\ Y_{1/2}^{1/2,-1/2} \end{pmatrix} = \begin{pmatrix} \sin \dfrac{\theta}{2} e^{i(\phi-\chi)/2} \\ \cos \dfrac{\theta}{2} e^{i(\phi+\chi)/2} \end{pmatrix}, \tag{21}$$

$$\varepsilon_2 = \begin{pmatrix} -Y_{1/2}^{-1/2,1/2} \\ Y_{1/2}^{-1/2,-1/2} \end{pmatrix} = \begin{pmatrix} -\cos \dfrac{\theta}{2} e^{-i(\phi+\chi)/2} \\ \sin \dfrac{\theta}{2} e^{-i(\phi-\chi)/2} \end{pmatrix}. \tag{22}$$

These are the orthonormal eigenvectors of an optical system M. Here a chiral photon is represented relativistically by two-component spinors which can be split as follows:

$$\begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix} = \begin{pmatrix} \psi_+ \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ \psi_- \end{pmatrix},$$

where the first and second terms represent the eigenstate of right circular polarization and left circular polarization, respectively.

With the help of Jones calculus, we want to construct the matrix of the optical system by considering the eigenvalue matrix as follows:

$$D = \begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix} = T^{-1} M T = \begin{pmatrix} 1/2 & 0 \\ 0 & -1/2 \end{pmatrix}, \tag{23}$$

which identifies the two helicities of polarized photons. Using the above equations, we can calculate the required polarization matrix as

$$M=TDT^{-1}=\begin{pmatrix} \sin\frac{\theta}{2}e^{i(\phi-\chi)/2} & -\cos\frac{\theta}{2}e^{-i(\phi+\chi)/2} \\ \cos\frac{\theta}{2}e^{i(\phi+\chi)/2} & \sin\frac{\theta}{2}e^{-i(\phi-\chi)/2} \end{pmatrix}$$
$$\times\begin{pmatrix} 1/2 & 0 \\ 0 & -1/2 \end{pmatrix}$$
$$\times\begin{pmatrix} \sin\frac{\theta}{2}e^{-i(\phi-\chi)/2} & \cos\frac{\theta}{2}e^{-i(\phi+\chi)/2} \\ -\cos\frac{\theta}{2}e^{i(\phi+\chi)/2} & \sin\frac{\theta}{2}e^{i(\phi-\chi)/2} \end{pmatrix},$$

$$M=\frac{1}{2}\begin{pmatrix} -\cos\theta & \sin\theta e^{-i\chi} \\ \sin\theta e^{i\chi} & \cos\theta \end{pmatrix}. \tag{24}$$

It can be verified that the above polarization matrix satisfies

$$M\varepsilon_1=1/2\varepsilon_1, \tag{25}$$

$$M\varepsilon_2=-1/2\varepsilon_2, \tag{26}$$

where eigenvalues reflect the helicity of polarized photons.

Rearranging the terms of Eq. (24), the polarization matrix M can have the representation

$$M=-\frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$
$$+\begin{pmatrix} \cos^2(\theta/2) & \sin(\theta/2)\cos(\theta/2)e^{-i\chi} \\ \sin(\theta/2)\cos(\theta/2)e^{i\chi} & \sin^2(\theta/2) \end{pmatrix}, \tag{27}$$

where we can identify the first term as the coherency matrix of natural radiation and the other term defines the cascaded optical system where a compensator is followed by a polarizer [20].

We now proceed to find the effect of a rotator introduced in the optical system such a way that it rotate the plane of polarization by an angle $\phi/2$ about the z axis. Then Eq. (19) helps us to write

$$M'=S(\phi/2)MS(-\phi/2), \tag{28}$$

which can be evaluated using Eq. (24),

$$M'=\frac{1}{2}\begin{pmatrix} \cos(\phi/2) & -\sin(\phi/2) \\ \sin(\phi/2) & \cos(\phi/2) \end{pmatrix}\begin{pmatrix} -\cos\theta & \sin\theta e^{-i\chi} \\ \sin\theta e^{i\chi} & \cos\theta \end{pmatrix}$$
$$\times\begin{pmatrix} \cos(\phi/2) & \sin(\phi/2) \\ -\sin(\phi/2) & \cos(\phi/2) \end{pmatrix}=\frac{1}{2}\begin{pmatrix} M'_{11} & M'_{12} \\ M'_{21} & M'_{22} \end{pmatrix}. \tag{29}$$

It has been found after calculation that

$$M'_{11}=-\cos\theta\cos\phi-\sin\phi\sin\theta\cos\chi,$$
$$M'_{12}=\sin\theta\cos\phi\cos\chi-\sin\phi\cos\theta-i\sin\theta\sin\chi,$$
$$M'_{21}=\sin\theta\cos\phi\cos\chi-\sin\phi\cos\theta+i\sin\theta\sin\chi,$$
$$M'_{22}=\sin\theta\sin\phi\cos\chi+\cos\theta\cos\phi. \tag{30}$$

It seems that the geometrical surface developed by the polarization matrix M' is parametrized by three variables θ, ϕ, and χ known as Euler angles. These type of single-gadget matrices M' belong to the SU(2) group. They have generated particular interest in the polarization optics as studied by Simon and Mukunda [9] as well as others. The usual polarization matrix (2×2) developed by the parameters θ and ϕ lies on the Poincaré sphere S^2, which is the spatial representation of the Mullar matrix representing the operation of optical activity in O_3. Thus matrix M' represented by Eq. (31) will belong to the extended Poincaré sphere where the additional parameter χ represents the angle which measures the anisotropy μ through the relation $\mu=i\ \delta/\delta\chi$. Our previous discussions (Sec. II) suggest that this variation of angle χ is also associated with the change of direction of helicity of the polarized photon. It can be realized from Eq. (9) that μ as well as χ is associated with a change of angular momentum. We can thus add here that a change of polarization of light is in connection with the change of angle χ which is associated with a transfer of angular momentum between the optical system and the incident light. This idea supports the analysis given by Tiwari [21].

The change of polarization matrix from M to M' will result in the emergent light differing from the initial state and it will satisfy in the quantum-mechanical framework the relation

$$|\psi_f\rangle=M'|\psi_i\rangle. \tag{31}$$

As the plane of polarization of the initial quantum-mechanical state $|\psi_i\rangle$ (two-component spinors) is made to rotate over a closed path, then apart from the dynamical phase there may appear the geometric phase of Berry. With this idea, the total phase change through the parallel transport of the polarization plane over a closed path becomes [22]

$$\langle\psi_f|\psi_i\rangle=e^{i(\gamma_d+\gamma_c)}. \tag{32}$$

Here γ_d is the dynamical phase and γ_c is the nontrivial phase associated with the geometry of the path. This celebrated geometrical phase $\gamma_c=-\mathrm{Im}\oint\langle n|dn\rangle$ originates from the eigenstates $|n\rangle$ which gives rise the curvature two-form.

To find the geometrical phase in a most simple and general way, we have chosen a point B on the closed path, which is developed by the rotator of rotational angle $\phi/2$ at fixed θ.

Let us first imagine the passage of circularly polarized light through a rotator. Then the polarization matrix and the state will belong to either of the two poles. It can be easily realized that rotation over a closed path will not trace any closed curve except the pole point. Yet in connection with the physical rotation of the angle $\phi/2$, we have the following state and the matrices representing the coordinates of the points $A(0,0)$ and $B(0,\phi)$ on the upper pole at $\theta=0$:

$$M'_A=\frac{1}{2}\begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix} \tag{33}$$

and

$$M'_B=\frac{1}{2}\begin{pmatrix}\cos\phi & -\sin\phi\\ -\sin\phi & \cos\phi\end{pmatrix}, \tag{34}$$

with

$$|\psi_A\rangle=\begin{pmatrix}0\\ e^{i\chi/2}\end{pmatrix}, \tag{35}$$

and from $|\psi'_A\rangle=M'_A M'_B|\psi_A\rangle$ we can find the total phase over the path ABA, which is

$$\langle\psi_A|\psi'_A\rangle=\frac{1}{4}\cos\phi. \tag{36}$$

In similar manner at the lower pole at $\theta=\pi$ apart, the coordinates of A and B are $(\pi,0)$ and (π,ϕ), for which the respective matrices are

$$M'_A=\frac{1}{2}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}, \tag{37}$$

$$M'_B=\frac{1}{2}\begin{pmatrix}\cos\phi & \sin\phi\\ \sin\phi & -\cos\phi\end{pmatrix}, \tag{38}$$

having

$$|\psi_A\rangle=\begin{pmatrix}e^{-i\chi/2}\\ 0\end{pmatrix}. \tag{39}$$

The required phase becomes

$$\langle\psi_A|\psi'_A\rangle=\frac{1}{4}\cos\phi. \tag{40}$$

It can be noted that the path ABA does not enclose any area. The appearance of the geometric phase is possible during the cyclic evolution if there is curvature associated with the closed path. Over an angle π, the phase 1/4 acquired by both the left or right circularly polarized light 2π will be off only the dynamical origin.

At the end, we shall consider the passage of linearly polarized light through a rotator rotated by an angle $\phi/2$. On an equitorial circle the value of points A and B are, respectively, $(\pi/2,0)$ and $(\pi/2,\phi)$. The corresponding matrices at those points and the initial states are

$$M'_A=\frac{1}{2}\begin{pmatrix}0 & e^{-i\chi}\\ e^{i\chi} & 0\end{pmatrix}, \tag{41}$$

$$M'_B=\frac{1}{2}\begin{pmatrix}-\sin\phi\,\cos\chi & \cos\phi\,\cos\chi-i\,\sin\chi\\ \cos\phi\,\cos\chi+i\,\sin\chi & \sin\phi\,\cos\chi\end{pmatrix}, \tag{42}$$

and

$$|\psi_A\rangle=\frac{1}{\sqrt{2}}\begin{pmatrix}e^{-i\chi/2}\\ e^{i\chi/2}\end{pmatrix}. \tag{43}$$

The phase developed by incident polarized light after a rotation of angle $\phi/2$ becomes,

$$\langle\psi_A|M'_A M'_B|\psi_A\rangle=\frac{1}{8}\,(e^{i\chi/2}\quad e^{-i\chi/2})\begin{pmatrix}0 & e^{-i\chi}\\ e^{i\chi} & 0\end{pmatrix}\begin{pmatrix}-\sin\phi\,\cos\chi & \cos\phi\,\cos\chi-i\,\sin\chi\\ \cos\phi\,\cos\chi+i\,\sin\chi & \sin\phi\,\cos\chi\end{pmatrix}\begin{pmatrix}e^{i(-\chi)/2}\\ e^{i\chi/2}\end{pmatrix}. \tag{44}$$

Through proper matrix multiplication and after simplifing, we can find the result

$$\langle\psi_i|\psi'_f\rangle=\frac{1}{8}\,[\cos 2\chi(1+\cos\phi)-(1-\cos\phi)]. \tag{45}$$

Here our required geometric phase for linearly polarized light will be identified as $\frac{1}{4}\cos 2\chi$, which is associated with the extra variable χ introduced for helicity of photon.

Our above results extends the idea of Jones [16], where he pointed out that a rotator does not change the state of polarization of circularly polarized light, whereas the reverse thing happens for plane-polarized light. It implies that if rotation over a closed path produces any change in the plane of polarization, the inclination of helicity (χ) will change and the developed phase will be of geometrical origin. This idea is visualized in our present work through the appearance of dynamical phase 1/4, for the passage of two opposite circularly polarized lights through the rotator. On the other hand, for the passage of plane-polarized light through a rotator, an additional phase $\cos 2\chi$ appears which is of geometric origin.

IV. DISCUSSION

In the light of Jones calculus, we have mathematically determined the polarization matrix of the optical system in the relativistic framework. We have chosen the polarized photon as a two-component spinor where the helicity is depicted by the extra variable χ. This causes us to parametrize our resultant polarization matrix by three variables θ, ϕ, and χ that lie on the extended Poincaré sphere.

In fact, we have shown finally that circularly polarized light possesses no geometrical phase (GP), only a dynamical phase (DP) after passage through rotator, whereas passage of plane polarized light able to produce a GP as well as a DP over the closed path. This study supports the idea of Jones [17] where appearance of a GP ensures the change of plane of polarization over a closed path.

In this connection, recently Berry has shown that the phase change for the light beam cycled by the twisted stack of N polarizers P is geometric, whereas for retarders R the corresponding phase change becomes geometric+dynamic [23]. These findings are consistent with our results as shown here.

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